# Maple Exploration of Hilbert Geometry in a Triangle

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***ABSTRACT***

*In this paper we explore Hilbert geometry in a triangle, using Maple to illustrate some concepts such as Hilbert distance, projective and affine coordinates, unitary circle, etc. and to introduce new "trigonometric functions" for this geometry.*

## 1. Introduction.

### 1.1 Some historical facts and context.

Euclid wrote his famous *Elements* around 300 B. C. In this thirteen - volume work, he brilliantly organized and presented the fundamental propositions of Greek geometry and number theory. In the first book of the Elements, Euclid develops plane geometry starting with five postulates, the first four of which never aroused controversy. However, the status of the fifth axiom (the so- called parallel axiom) was less clear, and became the subject of investigations. For many years mathematicians tried to prove the fifth postulate, in particular Gauss in 1792 discovered that the denial of the fifth postulate leads to a new strange geometry which he called non- Euclidean geometry. Only a few years passed before non- Euclidean geometry was rediscovered independently by Nikolai Lobachevsky and János Bolyai. Today, the non-Euclidean geometry of Gauss, Lobachevsky and Bolyai is called hyperbolic geometry, and the term non- Euclidean refers to any geometry that is not Euclidean.
Although Euclidean geometry is a fantastic accurate theory of space it is not, however, a perfect theory: modern experiments have reveled extremely small discrepancies between predictions of Euclidean geometry and the measured geometric properties of figures constructed in physical space. These departures from Euclidean geometry are now known to be governed in precise mathematical way, by the distributions of matter and energy in space. This is the essence of revolutionary theory of gravity discovered by Einstein in 1915.
Riemann highlighted the existence of three types of geometry: Euclidean, Hyperbolic and Elliptic. In his *Habilitationsvortrag* of 1854, Riemann introduced a metric structure in a general space based on the element of arc: $ds = F\left(x^1,\ldots x^n; dx^1 \ldots dx^2\right)$. Here, $F\left(x;y\right)$ is a positive (when $y \neq 0$) function on the tangent bundle $TM$ and is homogeneous of degree one in $y$. An important special case is when $F^2 = g_{ij}\left(x\right) dx^i dx^j$

Historical developments have conferred the name Riemannian geometry to this case while the general case, Riemannian geometry without the quadratic restriction, has been known as Finsler geometry [24]. The name "Finsler geometry" came from Finsler's thesis of 1918. It is actually the geometry of a simple integral and is as old as the calculus of variations. Hilbert attached great importance to the field, and in his famous Paris address of 1900 devoted Problem 23 to the variational calculus of $\int ds$ and its geometrical overtones.

On the other hand in 1870 Felix Klein produced an account that unified a large class of different geometries. According to Klein – Cayley there are at least nine different geometries in the plane, three of which are the previous mentioned Euclidean, Hyperbolic and Elliptic. The Klein – Cayley classification is based in the different ways that is possible to define distances between two points and the measure of the angle between two straight lines. This classification allowed Klein to create the so-called *Erlanger Program*. In his Erlanger Program Klein proposed that Euclidean and non- Euclidean geometry be regarded as special cases of projective geometry. In each case the common features that, according to Klein, made them geometries were that there were a set of points, called a "space," and a group of transformations by means of which figures could be moved around in the space without altering their essential properties. Different geometries would have different spaces and different groups, and the figures would have different basic properties.

In 1895 David Hilbert presented in his *Grundlagen der Geometrie* [1] a way of not merely sorting out the geometries in Klein's hierarchy according to the different axiom systems they obeyed but to new geometries as well. For the first time there was a way of discussing geometry that lay beyond even the very general terms proposed by Riemann.

In spite of not all of these different geometries have continued to be of interest for years, there has recently been growing interest in Hilbert Geometry [1] and many research papers were published, see in particular the papers by Benoist [2 - 6], Colbois, Verovic and Vernicos [7 - 11], Fortsch, Karlsson and Noskov [12 - 13], de la Harpe [14], the thesis of Socie-Methou [15 - 16] and the book by Chern and Shen [17] to cite just a few of them.

Mathematicians could ask why they had believed for so many years that Euclidean geometry to be the only one when, in fact, many different geometries existed. The German mathematician Moritz Pasch argued in 1882 that perhaps the mistake had been to rely too heavily on physical intuition. In his view an argument in mathematics and in particular geometry should depend for its validity not on the physical interpretation of the terms involved but upon purely formal.

Perhaps new cosmological discovers and models relating the destiny of the universe to its geometry brought new interest to different kind of geometry including the Hilbert's one. Hilbert geometry has unusual properties that can be used to explore the notion of geometry itself. Hilbert argued that the rules governing the use of mathematical terms were arbitrary, and each mathematician could choose them at will, provided only that the choices made were self-consistent. A mathematician produced abstract systems unconstrained by the needs of science, and, if scientists found an abstract system that fit one of their concerns, they could apply the system secure in the knowledge that it was logically consistent. Points of view like this one produced that the hegemony of Euclidean geometry was challenged by non- Euclidean geometry and projective geometry by 19 century.

**1.2 The Hilbert metric.**

The Hilbert metric is a canonical metric associated to an arbitrary bounded convex domain of $\mathbb{R}^n$. It has been proposed by David Hilbert as an example of a metric for which the Euclidean straight lines are shortest geodesic curves.

Let $H$ be a nonempty bounded open convex set in $\mathbb{R}^n$, with $n \geq 2$. The Hilbert distance "$distH$" on $H$ was introduced by D. Hilbert as follows. For any $P \in H$, let $distH(P,P) = 0$.

For distinct points $P$ and $Q$ in $H$, assume the line passing through $P, Q$ intersects the boundary $\partial\overline{H}$ at two points $X, Y$ such that the order of these four points on the line is $Y, P, Q, X$, see figure 1.

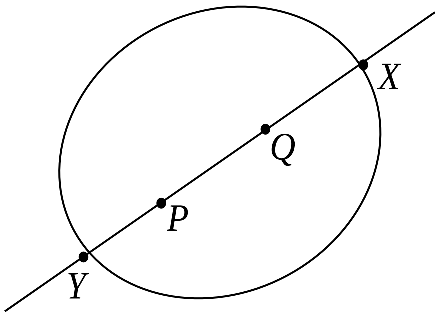



Figure 1

Denote the cross-ratio of these points by:

$$(PQXY) = \frac{\overline{PX}}{\overline{XQ}} : \frac{\overline{PY}}{\overline{YQ}}$$

Where the bar on letters means ordinary Euclidean distance on $\mathbb{R}^n$. Then the Hilbert distance is defined by:

$$distH(P,Q) = \left|\log(PQXY)\right| \qquad (1.1)$$

This is a well defined distance under which the points at boundary are "at infinite".

The metric space $(H, distH)$ is called Hilbert geometry. When $H$ is the unit open ball $\left\{(x_1, x_2, \ldots x_n) \in \mathbb{R}^n, \sum_{i=1}^{n} x_i^2 < 1\right\}$, $(H, distH)$ is the Klein model for the hyperbolic geometry.

Some important properties of formula (1.1) are:

i) The formula (1.1) is indeed a metric.
ii) This metric is Finslerian, provided the boundary of $H$ is smooth enough.
iii) The metric is projective, that is, the Euclidean straight lines are geodesic.

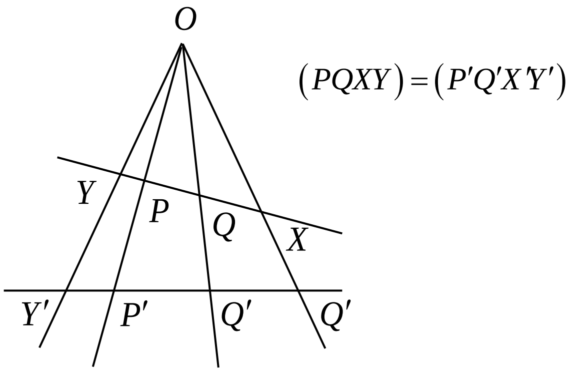



Figure 2

## 2. Distance and coordinates in Hilbert geometry in a triangle.

To explore Hilbert geometry in a triangle it will be convenient to introduce alternative definition for the Hilbert metric. Since the cross-ratio is invariant under any projective mapping T, $(H, distH)$ and $(T(H), distT(H))$ are isometric as Hilbert geometry (figure 2). We will use this property of the cross-ratio to introduce coordinates for points in $(H, distH)$ when $H$ is a triangle.
We know that given two points $A$ and $B$ in a segment, then a point $Q$ divides this segment in a ratio $k$ if $\frac{\overline{AQ}}{\overline{QB}} = k$. Solving for $Q$ we have $Q = a \cdot A + b \cdot B$
with $a + b = 1$. Where $a = \frac{1}{1+k}$ and $b = \frac{k}{1+k}$.
These relations are independent of the chosen origin of coordinates.
In the same way, given the vertices $A$, $B$ and $C$ of a triangle $H$, a point $P$ inside of it can be written as:

$$P = a \cdot A + b \cdot B + c \cdot C, \; a + b + c = 1, \; a \geq 0, b \geq 0, c \geq 0$$

We will call $(a, b, c)$ the ***projective coordinates*** of a point $P$.
Using these coordinates we can see, for example that points with $(0, b, c)$ correspond to the side $\overline{BC}$ etc.
Let be $P_C$ the projection of a point $P = (a, b, c)$ onto the side $\overline{AB}$ then:

$$P_C = \frac{a}{1-c} A + \frac{b}{1-c} B$$

The point $P_C$ divides the segment $\overline{AB}$ in a ratio $k = \frac{b}{a}$ (figure 3).

PROPOSITION 1: If $P=(a_1,b_1,c_1)$ and $Q=(a_2,b_2,c_2)$ are two points inside the triangle H and the straight line that joins them intersect the sides $\overline{AC}$ and $\overline{BC}$ then:

$$distH(P,Q)=\left|\log\left(\frac{b_2}{a_2}\cdot\frac{a_1}{b_1}\right)\right|. \qquad (2.1)$$

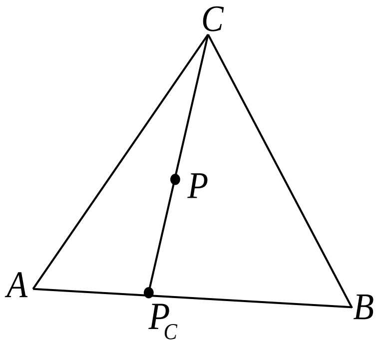



Figure 3

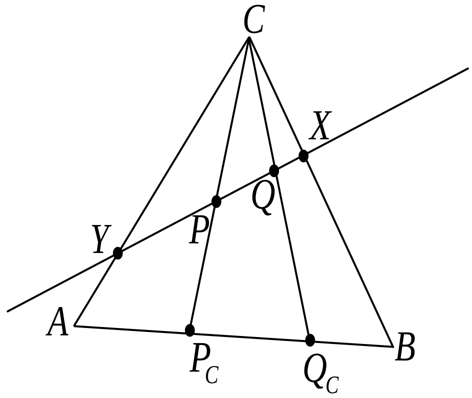



Figure 4

***Proof:***

By projection from C we will have (figure 4):

$$distH(P,Q)=\left|\log(PQXY)\right|=\left|\log(P_CQ_CBA)\right|=\left|\frac{\overline{P_CB}}{\overline{BQ_C}}:\frac{\overline{P_CA}}{\overline{AQ_C}}\right|$$

But $\dfrac{\overline{P_CB}}{\overline{P_CA}}=\dfrac{1}{k}=\dfrac{a_1}{b_1}$ and $\dfrac{\overline{AQ_C}}{\overline{BQ_C}}=l=\dfrac{b_2}{a_2}$ so, $distH(P,Q)=\left|\log\left(\dfrac{b_2}{a_2}\cdot\dfrac{a_1}{b_1}\right)\right|$ ■

In the same way, if the straight line that joins points $P=(a_1,b_1,c_1)$ and $Q=(a_2,b_2,c_2)$ intersects the sides $\overline{AB}$ and $\overline{BC}$ then:

$$distH(P,Q)=\left|\log\left(\frac{a_1}{c_1}\cdot\frac{c_2}{a_2}\right)\right| \qquad (2.2)$$

If the straight line joining these points intersects the sides $\overline{AB}$ and $\overline{AC}$ then:

$$distH(P,Q)=\left|\log\left(\frac{b_1}{c_1}\cdot\frac{c_2}{b_2}\right)\right| \qquad (2.3)$$

To explore the metric space $(H,distH)$ we construct the Maple procedure “disT” (see appendix A), which consider all these cases.

PROPOSITION 2: If $P=(a_1,b_1,c_1)$ and $Q=(a_2,b_2,c_2)$ are two points inside the triangle $H$ There exists a point $Z=(a_3,b_3,c_3)\in H$ such that $P$, $Q$ and $Z$ are not collinear and:

$$distH(P,Q)=distH(P,Z)+distH(Z,Q)$$

Using the procedures "disT" (appendix A) and "ttn" (appendix B) we colored the inside of the triangle $H$ according to the value of the function:

$$distH(P,Z)+distH(Z,Q)-distH(P,Q)=\Phi(Z) \qquad (2.4)$$

The results can be seen in the figure 5. Inside the quadrilateral represented in the figure, we have $\Phi(Z)=0$.

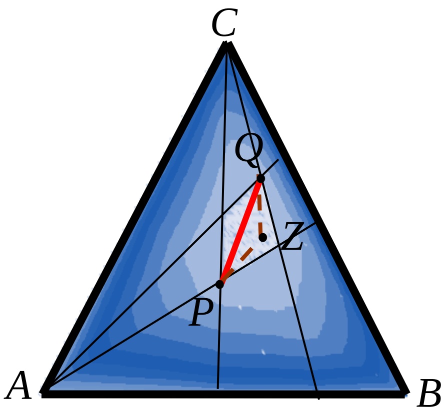



Figure 5

***Proof:***

Taking $Z$ inside the quadrilateral as it is shown in the figure we have:

$$distH(P,Z)+distH(Z,Q)=\left|\log\left(\frac{a_1}{c_1}\cdot\frac{c_3}{a_3}\right)\right|+\left|\log\left(\frac{a_3}{c_3}\cdot\frac{c_2}{a_2}\right)\right|=$$

$$=\log\left(\frac{a_1}{c_1}\cdot\frac{c_3}{a_3}\right)+\log\left(\frac{a_3}{c_3}\cdot\frac{c_2}{a_2}\right)=\log\left(\frac{a_1}{c_1}\right)+\cancel{\log\left(\frac{c_3}{a_3}\right)}+\cancel{\log\left(\frac{a_3}{c_3}\right)}+\log\left(\frac{c_2}{a_2}\right)=$$

$$\log\left(\frac{a_1}{c_1}\cdot\frac{c_2}{a_2}\right)=\left|\log\left(\frac{a_1}{c_1}\cdot\frac{c_2}{a_2}\right)\right|=distH(P,Q)$$

If $Z$ is outside of the quadrilateral then the distance must be calculated using different formulas because the straight line joining the points $P,Z$ and $Q,Z$ intersects different pairs of sides of the triangle ■

*This is an unusual property of the Hilbert geometry in a triangle, and means that there is not a unique geodesic line joining two different points.*

Together with the projective coordinates we will consider the ***affine coordinates*** of a point $P$. If $P=(a,b,c)$ with $a+b+c=1$ then affine coordinates for $P$ are:

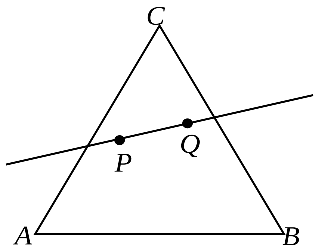
$$P = [x,y] \quad \text{with} \quad \begin{cases} x = a/c \\ y = b/c \end{cases} \qquad (2.5)$$

We suppose $P$ is strictly at the interior of the triangle, where a, $b$, $c$ are positives.
If we know affine coordinates $[x,y]$ then we can obtain the projective coordinates by the following formulas:

$$\begin{cases} a = \dfrac{x}{(1+x+y)} \\ b = \dfrac{y}{(1+x+y)} \\ c = \dfrac{1}{(1+x+y)} \end{cases} \qquad (2.6)$$

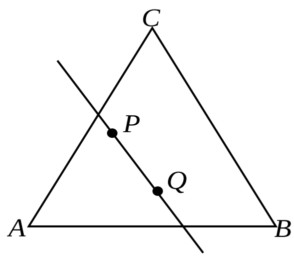
These affine coordinates has the advantage to be two (and not three as projective coordinates) in a space of dimension two.

The formula for distance "distH" in these coordinates however varies and we write it down here:

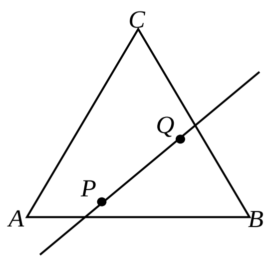



$$P=[x_1,y_1] \qquad Q=[x_2,y_2]$$

$$distH(P,Q)=\left|\log\left(\frac{y_2}{x_2}\frac{x_1}{y_1}\right)\right| \qquad (2.7)$$




$$distH(P,Q)=\left|\log\left(\frac{y_1}{y_2}\right)\right| \qquad (2.8)$$



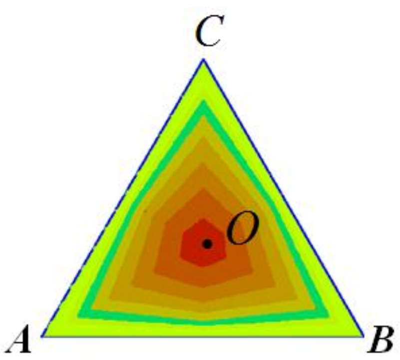

$$distH(P,Q)=\left|\log\left(\frac{x_1}{x_2}\right)\right| \qquad (2.9)$$

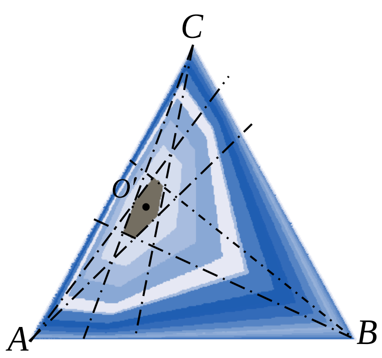
For example, the affine coordinates of the points in figure 8 for $r=1$ are:

$O=[1,1]$; $P_1=[1,e]$; $P_2=\left[e^{-1},1\right]$; $P_3=\left[e^{-1},e^{-1}\right]$; $P_4=\left[1,e^{-1}\right]$; $P_5=[e,1]$; $P_6=[e,e]$.

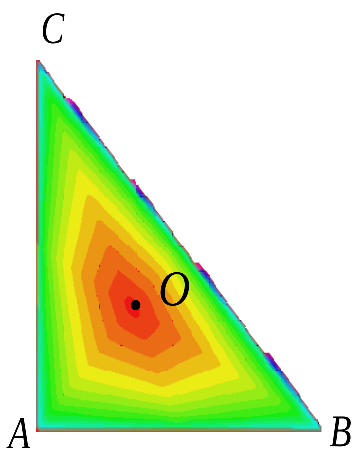
## 3. Circles and Disks.

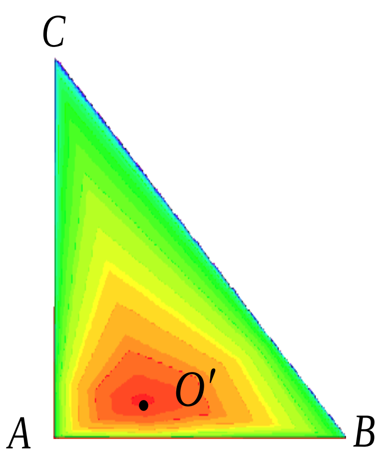
For our Maple explorations of Hilbert geometry we will use an equilateral triangle with vertices at points $A=(0,0)$, $B=(1,0)$ and $C=\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)$. Using the procedure "disT" (appendix A) together with the procedure "tt2" (appendix C) we explore the shapes of disks centered at the points $O=\frac{1}{3}\cdot A+\frac{1}{3}\cdot B+\frac{1}{3}\cdot C=\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$ (see figure 6) and $O'=\frac{3}{7}\cdot A+\frac{1}{8}\cdot B+\frac{25}{56}\cdot C=\left(\frac{3}{7},\frac{1}{8},\frac{25}{56}\right)$ (see figure 7) with different radius. Note that the "circles" are really hexagons.




This property is independent of the used triangle. If we use another triangle for example, with vertices at points $A=(0,0)$, $B=(1,0)$ and $C=(0,1)$ we obtain again "circles-hexagons" as in the following figures:



We can parameterize "circles" (boundary of the hexagons) centered at $O$ and radius $r$ using the following formulas (see figure 8):

$$\begin{aligned}
\overline{P_1P_2} &= \left(t, e^r \cdot t, 1-t-e^r \cdot t\right); & \overline{P_2P_3} &= \left(t, 1-t-e^r \cdot t, e^r \cdot t\right) \\
\overline{P_3P_4} &= \left(1-t-e^r \cdot t, t, e^r \cdot t\right); & \overline{P_4P_5} &= \left(e^r \cdot t, t, 1-t-e^r \cdot t\right) \\
\overline{P_5P_6} &= \left(e^r \cdot t, 1-t-e^r \cdot t, t\right); & \overline{P_6P_1} &= \left(1-t-e^r \cdot t, e^r \cdot t, t\right)
\end{aligned} \tag{3.1}$$

with $\dfrac{1}{2 \cdot e^r + 1} \le t \le \dfrac{1}{e^r + 2}$

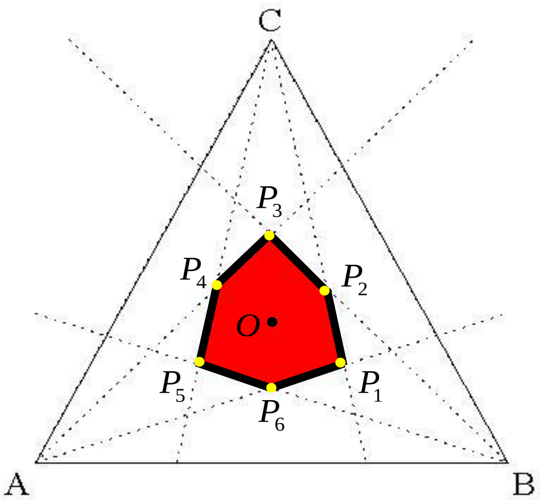



Figure 8

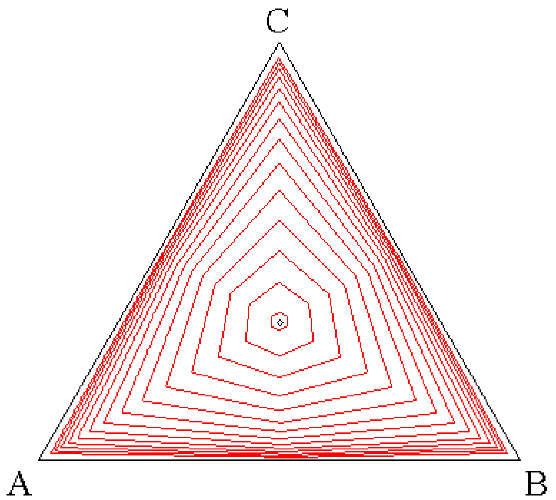



Figure 9

Using these formulas with Maple commands given in the appendix D we obtain circles with center $O$ and radius $r$ as it is shown in figure 9.

## 4. Measure of the angles and Trigonometric Functions.

Consider now the following definition of an angle for the Hilbert geometry in a triangle. Let's take two semi-straight lines that intersect each other in a vertex $P$. By an isometry we can take $P$ to the central point $O$; the semi-straight lines intersect the unitary circle in two points defining an arch in the circle. The length of this arch will be the measure of the angle.

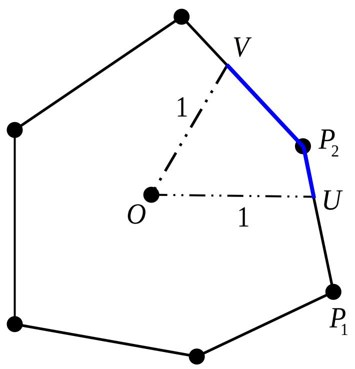



Figure 10

$$\alpha = \angle UOV = distH(U, P_2) + distH(P_2, V) \qquad (4.1)$$

It is easy to show that $\sum_{i=1}^{5} distH(P_i, P_{i+1}) + distH(P_6, P_1) = 6$ so the complete angle measures 6.

As in the classic case, we now consider a semi-straight line $OP_1$ that starts to move in a counter clockwise direction. We define $[C(\alpha), S(\alpha)]$ as the affine coordinates of the point $P$ if $\angle P_1OP = \alpha$, (figure 11).

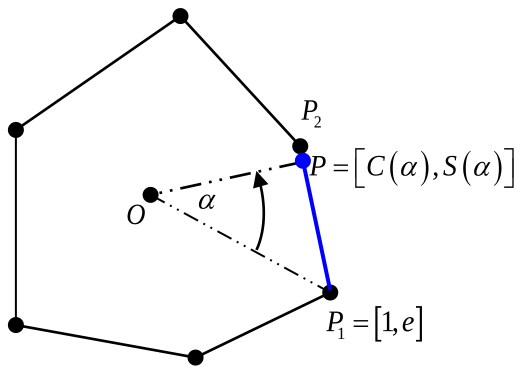



Figure 11

Using the procedure "Arch" (appendix E) we can determine the angle $\alpha$ for any point $P$ on the unitary circle given its affine coordinates. This procedure uses the formulas (3.1), automatically determines to which segment of the unitary circle $P$ belongs and applies the correct formula for the distance of $P$ from a point $P_1$. With this procedure we can construct the graphics of functions $C(\alpha)$ and $S(\alpha)$ as it is shown in figures 12 and 13.

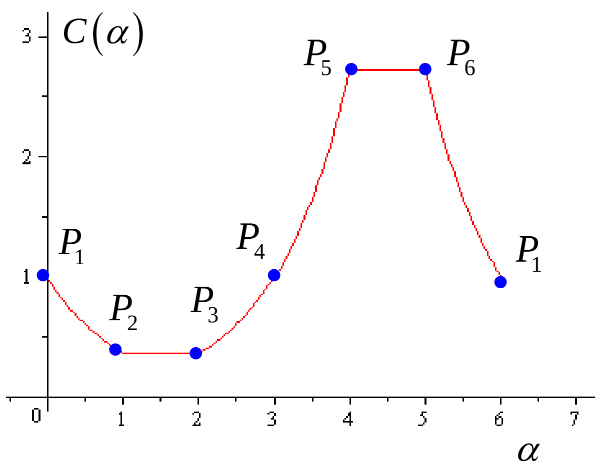



Figure 12

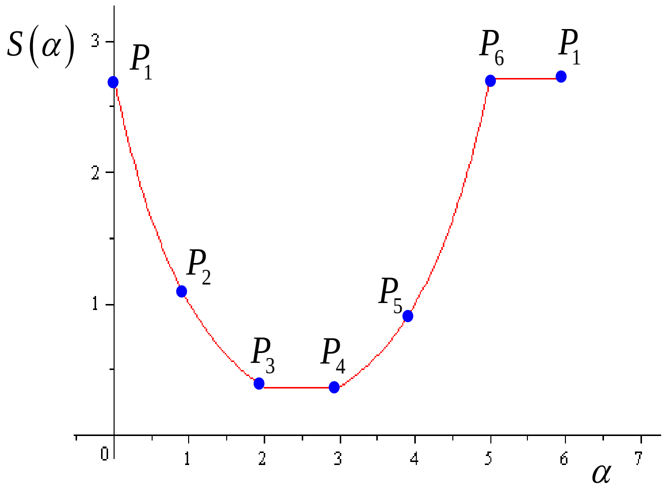



Figure 13

Moreover, these functions can be given in analytic form as it is shown in the following proposition.

PROPOSITION 3: The trigonometric functions take the following values:

$$\begin{cases} C(\alpha)=e^{-\alpha} \\ S(\alpha)=e^{1-\alpha} \quad \text{if} \quad 0\le\alpha\le 1 \end{cases} ; \quad \begin{cases} C(\alpha)=e^{-1} \\ S(\alpha)=e^{1-\alpha} \quad \text{if} \quad 1\le\alpha\le 2 \end{cases}$$

$$\begin{cases} C(\alpha)=e^{\alpha-3} \\ S(\alpha)=e^{-1} \quad \text{if} \quad 2\le\alpha\le 3 \end{cases} ; \quad \begin{cases} C(\alpha)=e^{\alpha-3} \\ S(\alpha)=e^{\alpha-4} \quad \text{if} \quad 3\le\alpha\le 4 \end{cases}$$

$$\begin{cases} C(\alpha)=e \\ S(\alpha)=e^{\alpha-4} \quad \text{if} \quad 4\le\alpha\le 5 \end{cases} ; \quad \begin{cases} C(\alpha)=e^{6-\alpha} \\ S(\alpha)=e \quad \text{if} \quad 5\le\alpha\le 6 \end{cases}$$

These formulas can be extended by periodicity for $\alpha \le 0$ or also for $\alpha \ge 6$, (see figures 14 and 15).

***Proof:***

We will verify for instance formula (1). By using different versions of distance (2.7 – 2.9) can be proved in the same way the rest of the formulas
Using formula 2.7 we have:

$$distH(O,P) \;=\; distH\left( [1,1],\ \left[e^{-\alpha},e^{1-\alpha}\right]\right)$$

$$= \left| \log\left(\frac{1}{1}\cdot\frac{e^{1-\alpha}}{e^{-\alpha}}\right) \right| = \log(e) = 1$$

Using formula (2.9)

$$distH\left(P, P_1\right) = distH\left(\left[1, e\right], \left[e^{-\alpha}, e^{1-\alpha}\right]\right)$$

$$= \left| \log\left(\frac{e^{-\alpha}}{1}\right) \right| = \log\left(e^{\alpha}\right) = \alpha \text{ . } \blacksquare$$

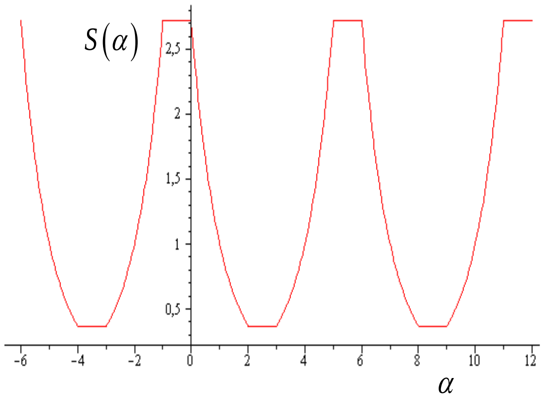



Figure 14

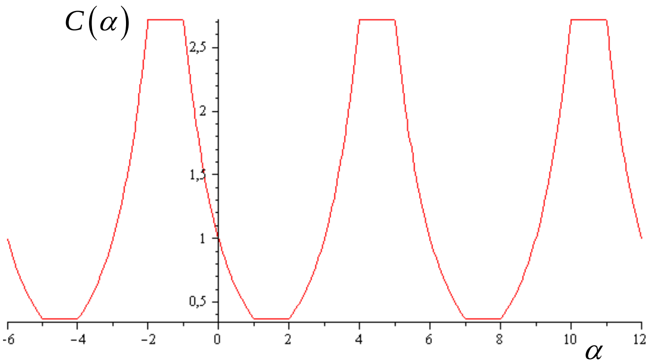



Figure 15

Using $S(\alpha)$ and $C(\alpha)$ we can define new functions:

$$T(\alpha) = \frac{S(\alpha)}{C(\alpha)} \; ; \qquad Ct(\alpha) = \frac{C(\alpha)}{S(\alpha)}.$$

The graphics of these functions are shown in figures 16 and 17 respectively.

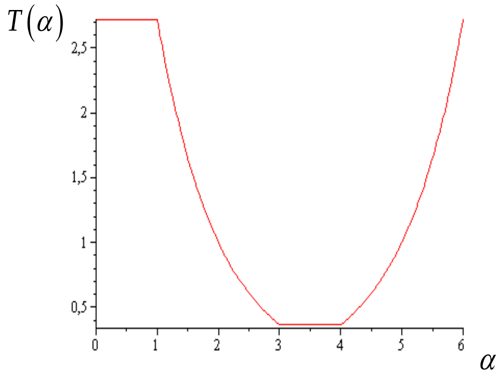



Figure 16

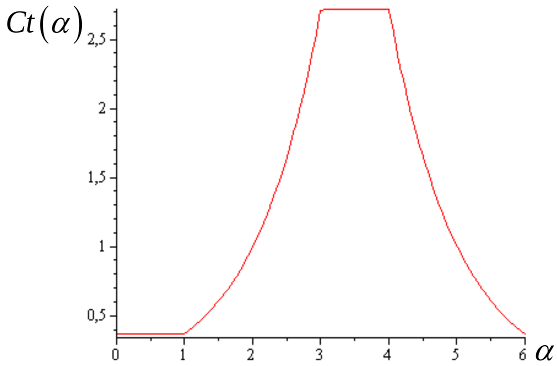



Figure 17

With the proposition 3 and the graphics of the functions $S(\alpha)$, $C(\alpha)$, $T(\alpha)$ and $Ct(\alpha)$ we can proof the following proposition:

PROPOSITION 4: The functions $S(\alpha)$, $C(\alpha)$, $T(\alpha)$ and $Ct(\alpha)$ satisfy the following relations:

$$C(\alpha-1)=S(\alpha);$$
$$C(\alpha+1)=Ct(\alpha)$$
$$S(\alpha-1)=T(\alpha);$$
$$Ct(\alpha-2)=T(\alpha+1)$$

## 5. Alternative definitions for trigonometric functions.

With the view to gaining more familiar properties of trigonometric functions for Hilbert geometry in a triangle, we could adopt the following alternative definitions:

$$s(\alpha)=\ln(S(\alpha));$$
$$c(\alpha)=\ln(C(\alpha))$$

$$t(\alpha) = \frac{s\left(\alpha - \tfrac{1}{2}\right)}{c(\alpha)} = \frac{\ln\left(S\left(\alpha - \tfrac{1}{2}\right)\right)}{\ln\left(C(\alpha)\right)};$$

$$ct(\alpha) = \frac{c(\alpha)}{s\left(\alpha - \tfrac{1}{2}\right)} = \frac{\ln\left(C(\alpha)\right)}{\ln\left(S\left(\alpha - \tfrac{1}{2}\right)\right)}$$

The graphics for these functions are shown in the figures 18 - 21:

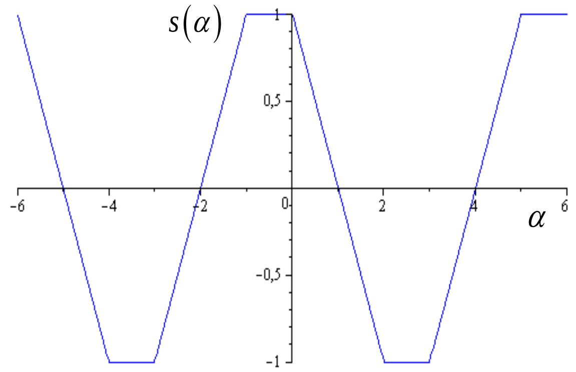



Figure 18

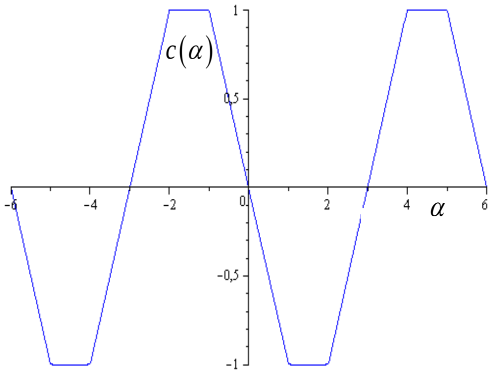



Figure 19

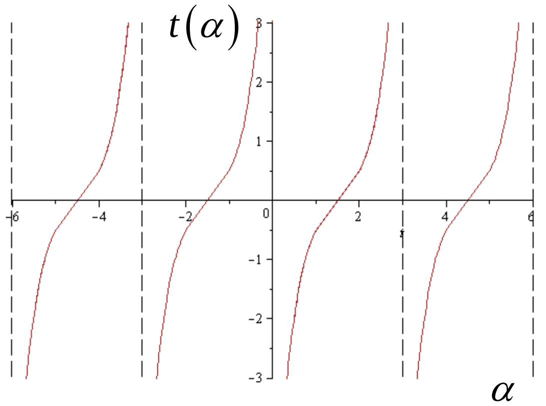



Figure 20

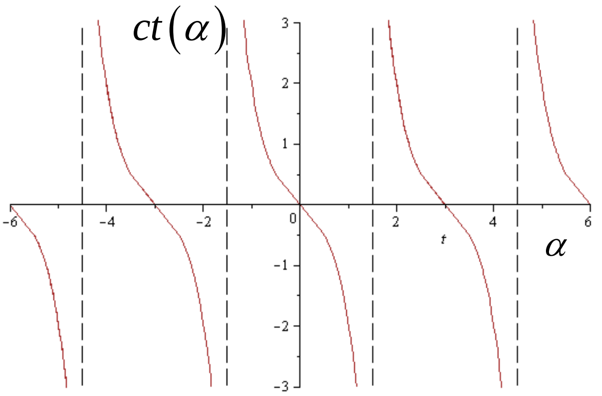



Figure 21

PROPOSITION 5: The functions $s(\alpha), c(\alpha)$, $t(\alpha)$ and $ct(\alpha)$ satisfy the following relations:

$$s(\alpha)=s(\alpha+6);$$
$$c(\alpha)=c(\alpha+6);$$
$$c(\alpha-1)=s(\alpha);$$
$$c(-\alpha)=-c(\alpha);$$
$$s\left(\alpha-\frac{1}{2}\right)=s\left(\frac{1}{2}-\alpha\right);$$
$$t(\alpha)=t(\alpha+3);$$
$$ct(\alpha)=ct(\alpha+3)\ .$$

## 6. Conclusions.

Hilbert Geometry is a particularly simple metric space on the interior of a compact convex set $H \in \mathbb{R}^n$ that can be used to explore and visualize properties of non-Euclidean and even non Riemannian geometry. It has been proposed by David Hilbert as an example of non- Euclidean geometry for which the Euclidean straight lines are shortest geodesic curves. Since the definition of the Hilbert geometry only uses cross ratios, the Hilbert metric is a projective invariant. It possesses a series of unusual properties that hits our common sense and physical intuition. In particular, the distance between two points depends of its relative positions in the space, (i. e. direction) this characteristic is common to finslerian metric, moreover in the case when the space $H$ is a simplex as the triangle in our paper there are more than one geodesic joining two distinct points and the “circles” are hexagons. In spite of these strange properties, we can define a measure of angles and trigonometric functions that have apparently similar properties than common trigonometric functions of Euclidean geometry. What results to be a little more complicated is define the area of a triangle, however some inside has been done recently in [25,26].

## 6. Supplementary Electronic Materials

Maple codes

## 7. References.

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## Appendix A

$disT := \mathbf{proc}(a1, b1, c1, a2, b2, c2)$
$\mathbf{local}\ r, s, p1, p2, A, B, C, d, P1, P2, P3:$
$with(VectorCalculus):$
$A := \langle 0, 0\rangle : B := \langle 1, 0\rangle : C := \left\langle \frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle :$
$p1 := a1 \cdot A + b1 \cdot B + c1 \cdot C :$
$p2 := a2 \cdot A + b2 \cdot B + c2 \cdot C :$
$\mathbf{if}\ (p2[2] = p1[2])\ \mathbf{then}\ \sqrt{\left(\ln\left(\frac{b2 \cdot a1}{a2 \cdot b1}\right)\right)^2}$
$\mathbf{elif}\ \left(p2[1] = p1[1]\ \mathbf{and}\ p1[1] \le \frac{1}{2}\right)\ \mathbf{then}\ \sqrt{\left(\ln\left(\frac{c2 \cdot b1}{b2 \cdot c1}\right)\right)^2}$
$\mathbf{elif}\ \left(p2[1] = p1[1]\ \mathbf{and}\ p1[1] \ge \frac{1}{2}\right)$
$\mathbf{then}\ \sqrt{\left(\ln\left(\frac{a1 \cdot c2}{c1 \cdot a2}\right)\right)^2}\ \mathbf{else}$
$r := \frac{p2[2] - p1[2]}{p2[1] - p1[1]} : s := p1[2] - p1[1] \cdot r :$
$P1 := evalb\left(\left(-\frac{s}{r} > 0\right)\ \mathbf{and}\ \left(-\frac{s}{r} < 1\right)\right) :$
$P2 := evalb\left(\left(\frac{\sqrt{3} - s}{\sqrt{3} + r} \ge \frac{1}{2}\right)\ \mathbf{and}\ \left(\frac{\sqrt{3} - s}{\sqrt{3} + r} < 1\right)\right) :$
$P3 := evalb\left(\left(\frac{s}{\sqrt{3} - r} > 0\right)\mathbf{and}\ \left(\frac{s}{\sqrt{3} - r} \le \frac{1}{2}\right)\right) :$
$\mathbf{if}\ (P1\ \mathbf{and}\ P2)\ \mathbf{then}\ \sqrt{\left(\ln\left(\frac{a1 \cdot c2}{c1 \cdot a2}\right)\right)^2}\ \mathbf{elif}\ (P1\ \mathbf{and}\ P3)$
$\mathbf{then}\ \sqrt{\left(\ln\left(\frac{c2 \cdot b1}{b2 \cdot c1}\right)\right)^2}\ \mathbf{else}\ \sqrt{\left(\ln\left(\frac{b2 \cdot a1}{a2 \cdot b1}\right)\right)^2}\ \mathbf{end\ if};$
$\mathbf{end\ if};$
$\mathbf{end}:$

## Appendix B

$ttn := \mathbf{proc}(x, y)$
$\mathbf{local}\ c, B, A, C, p, a, b :$
$a := evalf\left(-x - \frac{1}{3}\, y \cdot \sqrt{3} + 1\right) : b := evalf\left(x - \frac{1}{3}\, y \cdot \sqrt{3}\right) :$
$\mathbf{if}\ (evalf\,(disT(a1, b1, c1, a, b, 1 - a - b) + disT(a2, b2, c2, a, b, 1$
$- a - b) - disT(a1, b1, c1, a2, \mathrm{b2}, \mathrm{c2}))) < 10^{-9}\ \mathbf{then}\ (x)$
$\mathbf{else}\ evalf\,(disT(a1, b1, c1, a, b, 1 - a - b) + disT(a2, b2, c2,$
$a, b, 1 - a - b))\,\mathbf{end\ if};$
$\mathbf{end}:$

$nnn := plot3d\left(0, 0\,..\,1, 0\,..\,\frac{5 \cdot \sqrt{3}}{8}, orientation = [-90, 0], grid\right.$
$= [250, 250], style = patchnogrid,$
$\left. scaling = constrained, color = ttn\right) :$

## Appendix C

$tt2 := \mathbf{proc}(x, y)$
$\mathbf{local}\ c, B, A, C, p, a, b :$
$a := evalf\left(-x - \frac{1}{3}y\cdot\sqrt{3} + 1\right) : b := evalf\left(x - \frac{1}{3}y\cdot\sqrt{3}\right) :$
$disT\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, a, b, 1 - a - b\right);$
$\mathbf{end}:$

$nn := plot3d\left(0, 0 .. 1, 0 .. \frac{5\cdot\sqrt{3}}{8}, orientation = [-90, 0], grid\right.$
$= [200, 200], style = patchnogrid,$
$\left.scaling = constrained, color = tt2\right) :$

## Appendix D

> $restart :$
> $with(VectorCalculus) :$
> $A := \langle 0, 0\rangle : B := \langle 1, 0\rangle : C := \left\langle \frac{1}{2}, \frac{\sqrt{3}}{2}\right\rangle : di := 4 : step := 0.3 :$
> $P0 := \frac{1}{3}\cdot A + \frac{1}{3}\cdot B + \frac{1}{3}\cdot C :$
> $P1P2 := t\cdot A + e^{r}\cdot t\cdot B + (1 - t - e^{r}\cdot t)\cdot C :$
> $P2P3 := t\cdot A + (1 - t - e^{r}\cdot t)\cdot B + e^{r}\cdot t\cdot C :$
> $P3P4 := (1 - t - e^{r}\cdot t)\cdot A + t\cdot B + e^{r}\cdot t\cdot C :$
> $P4P5 := e^{r}\cdot t\cdot A + t\cdot B + (1 - t - e^{r}\cdot t)\cdot C :$
> $P5P6 := e^{r}\cdot t\cdot A + (1 - t - e^{r}\cdot t)\cdot B + t\cdot C :$
> $P6P1 := (1 - t - e^{r}\cdot t)\cdot A + e^{r}\cdot t\cdot B + t\cdot C :$
> $p1p2 := seq\left(plot\left(\left[P1P2[1], P1P2[2], t = \frac{1}{2\cdot e^{r} + 1} .. \frac{1}{2 + e^{r}}\right],\right.\right.$
> $\left.\left.color = red\right), r = 0.1 .. \mathrm{di}, \mathrm{step}\right) :$
> $p2p3 := seq\left(plot\left(\left[P2P3[1], P2P3[2], t = \frac{1}{2\cdot e^{r} + 1} .. \frac{1}{2 + e^{r}}\right],\right.\right.$
> $\left.\left.color = red\right), r = 0.1 .. di, step\right) :$
> $p3p4 := seq\left(plot\left(\left[P3P4[1], P3P4[2], t = \frac{1}{2\cdot e^{r} + 1} .. \frac{1}{2 + e^{r}}\right],\right.\right.$
> $\left.\left.color = red\right), r = 0.1 .. di, step\right) :$

> $p4p5 := seq\left(plot\left(\left[P4P5[1], P4P5[2], t = \frac{1}{2\cdot e^r + 1} .. \frac{1}{2 + e^r}\right], color = red\right), r = 0.1 .. di, step\right):$

> $p5p6 := seq\left(plot\left(\left[P5P6[1], P5P6[2], t = \frac{1}{2\cdot e^r + 1} .. \frac{1}{2 + e^r}\right], color = red\right), r = 0.1 .. di, step\right):$

> $p6p1 := seq\left(plot\left(\left[P6P1[1], P6P1[2], t = \frac{1}{2\cdot e^r + 1} .. \frac{1}{2 + e^r}\right], color = red\right), r = 0.1 .. di, step\right):$

> $p0 := \mathrm{plottools}[point]([P0[1], P0[2]]):$

> $l1 := plottools[line]([0, 0], [1, 0]):$

> $l2 := plottools[line]\left([0, 0], \left[\frac{1}{2}, \frac{\sqrt{3}}{2}\right]\right):$

> $l3 := plottools[line]\left([1, 0], \left[\frac{1}{2}, \frac{\sqrt{3}}{2}\right]\right):$

> $t1 := plots[textplot]\left(\left[[0, 0, \text{"A"}, font = [TIMES, ROMAN, 20], align = \{below, left\}], [1, 0, \text{"B"}, font = [TIMES, ROMAN, 20], align = \{below, right\}], \left[\frac{1}{2}, \frac{\sqrt{3}}{2}, \text{"C"}, font = [TIMES, ROMAN, 20], align = \{above\}\right]\right]\right):$

> $plots[display](\{p1p2, p2p3, p3p4, p4p5, p5p6, p6p1, p0, l1, l2, l3, t1\}, scaling = constrained, axes = none);$

## *Appendix E*

$Arch := \mathbf{proc}(x, y)$
$\mathbf{local}\ A, B, C, p1, p2, p3, p4, p5, p6, H, s1, s2, s3, s4, s5, s6, s, p, P12, P23, P34, P45, P56, P61, a, b, c :$
$with(VectorCalculus) :$

$$A := \langle 0, 0\rangle : B := \langle 1, 0\rangle : C := \left\langle \frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle :$$

$$p1 := evalf\left( \frac{1}{2 + \exp(1.0)} \cdot A + \frac{\exp(1.0)}{2 + \exp(1.0)} \cdot B + \frac{1}{2 + \exp(1.0)} \cdot C \right) :$$

$$p4 := evalf\left( \frac{\exp(1.0)}{1 + 2 \cdot \exp(1.0)} \cdot A + \frac{1.0}{1 + 2 \cdot \exp(1.0)} \cdot B + \frac{\exp(1.0)}{1 + 2 \cdot \exp(1.0)} \cdot C \right) :$$

$$p2 := evalf\left( \frac{1.0}{1 + 2 \cdot \exp(1.0)} \cdot A + \frac{\exp(1.0)}{1 + 2 \cdot \exp(1.0)} \cdot B + \frac{\exp(1.0)}{1 + 2 \cdot \exp(1.0)} \cdot C \right) :$$

$$p3 := evalf\left( \frac{1}{2 + \exp(1.0)} \cdot A + \frac{1.0}{2 + \exp(1.0)} \cdot B + \frac{\exp(1.0)}{2 + \exp(1.0)} \cdot C \right) :$$

$$p5 := evalf\left( \frac{\exp(1.0)}{2 + \exp(1.0)} \cdot A + \frac{1.0}{2 + \exp(1.0)} \cdot B + \frac{1.0}{2 + \exp(1.0)} \cdot C \right) :$$

$$p6 := evalf\left( \frac{\exp(1.0)}{1 + 2 \cdot \exp(1.0)} \cdot A + \frac{\exp(1.0)}{1 + 2 \cdot \exp(1.0)} \cdot B + \frac{1.0}{1 + 2 \cdot \exp(1.0)} \cdot C \right) :$$

$$\mathrm{H} := \frac{1}{3} \cdot A + \frac{1}{3} \cdot B + \frac{1}{3} \cdot C : \ s1 := evalf\left( \frac{p1[2] - \mathrm{H}[2]}{p1[1] - \mathrm{H}[1]} \right) : \ s2 := evalf\left( \frac{p2[2] - \mathrm{H}[2]}{p2[1] - \mathrm{H}[1]} \right) :$$

$$s4 := evalf\left( \frac{p4[2] - \mathrm{H}[2]}{p4[1] - \mathrm{H}[1]} \right) : \ s5 := evalf\left( \frac{p5[2] - \mathrm{H}[2]}{p5[1] - \mathrm{H}[1]} \right) :$$

$$a := evalf\left( \frac{x}{1 + x + y} \right) :$$

$$b := evalf\left( \frac{y}{1 + x + y} \right) : \quad c := evalf(1 - a - b) : p := a \cdot A + b \cdot B + c \cdot C :$$

$$s := evalf\left( \frac{p[2] - \mathrm{H}[2]}{p[1] - \mathrm{H}[1]} \right) : P12 := evalb((s < s2)\ \mathbf{and}\ (s \geq s1)\ \mathbf{and}\ (evalf(p[1]) > evalf(H[1]))) :$$

$$P23 := evalb((s \geq s2)\ \mathbf{and}\ (evalf(p[2]) > evalf(H[2]))\ \mathbf{and}\ (evalf(p[1]) \geq evalf(H[1]))) :$$

$$P34 := evalb(\ (\mathrm{abs}(s) > \mathrm{abs}(s4))\ \mathbf{and}\ (evalf(p[2]) \geq evalf(H[2]))\mathbf{and}\ (evalf(p[1]) \leq evalf(H[1]))) :$$

$$P45 := evalb((evalf(p[1]) < evalf(H[1]))\ \mathbf{and}\ (((s \geq 0)\ \mathbf{and}\ (s \leq s5))\mathbf{or}((s < 0)\ \mathbf{and}\ (\mathrm{abs}(s) \leq \mathrm{abs}(s4))))\ ) :$$

$$P56 := evalb((s \geq s5)\ \ \mathbf{and}\ (evalf(p[2]) < evalf(H[2]))) : P61 := evalb((s < s1)\ \ \mathbf{and}\ (evalf(p[2]) < evalf(H[2]))\ ) :$$

$\mathbf{if}\ (P12)\ \mathbf{then}\ (disTa(x, y, 1, evalf(\exp(1.0)))\ )\ \mathbf{elif}\ \ (P23)$
$\quad \mathbf{then}\ \ (disTa(x, y, evalf(\exp(-1.0)), 1.0) + 1)\ \mathbf{elif}\ \ (P34)$
$\quad \mathbf{then}\ \ \left( disTa\left( x, y, evalf\left( \frac{1}{\exp(1.0)} \right), evalf\left( \frac{1}{\exp(1.0)} \right) \right) + 2 \right)\ \ \mathbf{elif}\ \ (P45)\ \ \mathbf{then}\ \ \left( disTa\left( x, y, 1.0, evalf\left( \frac{1}{\exp(1.0)} \right) \right) + 3 \right)\ \ \mathbf{elif}\ \ (P56)\ \ \mathbf{then}\ \ (disTa(x, y, evalf(\exp(1.0)), 1.0) + 4\ )$
$\quad \mathbf{else}\ \ (disTa(x, y, evalf(\exp(1.0)), evalf(\exp(1.0))) + 5)$
**end if**;
**end**: